1. INTRODUCTION

The van der Waals (vdW) interaction of two neutral, unpolarized, but polarizable atoms is a well-known consequence of quantum ground-state fluctuations. For sufficiently small separations, its physical origin may be seen in the electrostatic Coulomb interaction of the atoms' fluctuating dipole moments. The vdW interaction was first calculated in this nonretarded limit by London on the basis of perturbation theory, who found an attractive potential proportional to $r^{-6}$, where $r$ denotes the interatomic separation [1]. For larger separations, the vacuum fluctuations of the (transverse) electromagnetic field also contribute to the interaction. This was first taken into account by Casimir and Polder by means of a normal-mode expansion of the electromagnetic field, generalizing the London potential to arbitrary distances and showing that in particular in the retarded limit the potential varies as $r^{-7}$ [2].

The theory has since been extended in many respects and various factors affecting the vdW interaction have been studied. It has been shown that in the case of one [3] or both atoms [4, 5] being excited, the vdW potential varies as $r^{-6}$ and $r^{-2}$ in the nonretarded and retarded limits, respectively. Thermal photons present for finite temperature have been found to lead to a change of the retarded vdW potential of two ground-state atoms from a $r^{-7}$- to a $r^{-6}$-dependence as soon as the interatomic separation exceeds the wavelength of the dominant photons [6–9]. The influence of external electric fields on the vdW interaction was addressed, where it has been found that the resulting potential varies as $r^{-3}$ in the nonretarded limit when the applied field is unidirectional [10]. Generalizations of the vdW interaction to the three- [11–14] and $N$-atom case [15, 16] were studied first in the nonretarded limit and later for arbitrary interatomic separations, where the potentials were seen to depend on the relative positions of the atoms in a rather complicated way.

The two-atom vdW interaction may be strongly affected by the presence of magnetoelectric bodies. This was first demonstrated by Mahanty and Ninham, who employed a semiclassical approach to obtain a general expression for the vdW potential of two ground-state atoms in the presence of electric bodies [17–19] and applied it to the case of two atoms placed between two perfectly conducting plates [18]. The situation of two atoms between two perfectly conducting plates was later reconsidered taking into account finite temperature effects [20]. Other scenarios, such as two atoms placed within a planar electric three-layer geometry [21] or two anisotropic molecules in front of an electric half space or within a planar electric cavity, have also been studied [22].

Bearing in mind that the vdW potential of two polarizable atoms may be modified due to finite temperature, external fields, or the presence of electric bodies, but remains attractive in all of these cases, it is rather surprising that the interaction of a polarizable atom with a
magnetizable one is repulsive. This was first realized by Feinberg and Sucher, who restricted their attention to the retarded case and found a repulsive potential proportional to \( r^{-7} \) [23]. Their result was later extended to all distances [24, 25] and, in particular, it was shown that the nonretarded potential is proportional to \( r^{-4} \) [26].

Van der Waals interactions of two atoms exhibiting electric, as well as magnetic, properties have so far only been studied in the free-space case. A much richer range of phenomena is to be expected when allowing for the presence of magnetoelectric bodies, where a complex interplay of the electric and magnetic properties of the atoms and the bodies influences both the sign and functional dependence of the two-atom vdW potential. This problem is addressed in the current work, where we derive the vdW potential of a polarizable atom with another polarizable or magnetizable one by starting from its Casimir–Polder (CP) interaction with a weakly polarizable or magnetizable body, respectively. This approach, which has the advantage of being much simpler than perturbative methods and easily applicable to magnetizable atoms, renders general expressions for the two-atom vdW potentials of polarizable and/or magnetizable atoms in the presence of an arbitrary arrangement of magnetoelectric bodies, as is shown in Section 2. In Section 3, the general results are applied to the vdW interaction between polarizable/magnetizable atoms in free space and in front of a magnetoelectric plate. A summary is given in Section 4.

2. GENERAL THEORY

Consider first a neutral, nonpolar, ground-state atom or molecule \( A \) (briefly referred to as atom in the following) in the presence of an arbitrary arrangement of dispersing and absorbing magnetoelectric bodies. The atom is characterized by its center-of-mass position \( r_A \) and its frequency-dependent electric polarizability \( \alpha_A(\omega) \), while the bodies are given by their macroscopic (relative) permittivity \( \varepsilon(r, \omega) \) and permeability \( \mu(r, \omega) \), which are spatially varying, complex-valued functions of frequency, with the corresponding Kramers–Kronig relations being satisfied.

Due to the presence of the bodies, the atom will be subject to a CP force \( \mathbf{F}_A \). Within the framework of macroscopic QED in linear causal media and by using leading-order perturbation theory, it can be shown that this force follows from the associated potential [27, 28]

\[
U_A(r_A) = \frac{\hbar |\mathbf{H}_A|}{2\pi} \int_0^\infty d u \, u^2 \alpha_A(iu) \text{Tr} \mathbf{G}^{(1)}(r_A, r_A, iu)
\]

(1)

according to

\[
\mathbf{F}_A = -\nabla_A U_A(r_A)
\]

(2)

For \( \nabla_A = \nabla_{r_A} \). In Eq. (1), \( \mathbf{G}^{(1)}(r, r', \omega) \) is the scattering part of the classical Green tensor of the electromagnetic field,

\[
\mathbf{G}(r, r', \omega) = \mathbf{G}^{(0)}(r, r', \omega) + \mathbf{G}^{(1)}(r, r', \omega)
\]

(3)

\( \mathbf{G}^{(0)}(r, r', \omega) \) (bulk part), which is the solution to the equation

\[
\left[ \nabla \times \kappa(r, \omega) \nabla \times -\frac{\omega^2}{c^2} \varepsilon(r, \omega) \right] \mathbf{G}(r, r', \omega) = \delta(r - r')
\]

(4)

(\( \kappa(r, \omega) = \mu^{-1}(r, \omega) \)) together with the boundary condition

\[
\mathbf{G}(r, r', \omega) \rightarrow 0 \quad \text{for} \quad |r - r'| \rightarrow \infty.
\]

(5)

In order to derive the vdW interaction of atom \( A \) with a second polarizable atom \( B \), we now introduce an additional weakly polarizable body of (small) volume \( V_b \), consisting of a collection of atoms of type \( B \). Provided that the atomic number density \( \eta_b(r) \) is sufficiently small, the electric susceptibility of the additional body can be approximated by

\[
\chi_e(r, \omega) = \begin{cases} \varepsilon_0^{-1} \eta_b(r) \alpha_b(\omega) & \text{if } r \in V_b \\ 0 & \text{if } r \not\in V_b, \end{cases}
\]

(6)

so that the permittivity of the total arrangement of bodies reads \( \varepsilon(r, \omega) + \chi_e(r, \omega) \), and the corresponding Green tensor is given by Eq. (4) with \( \varepsilon(r, \omega) + \chi_e(r, \omega) \) instead of \( \varepsilon(r, \omega) \). A linear expansion of this differential equation in terms of \( \chi_e \) reveals that the presence of the additional body leads to a change of the Green tensor, whose leading term is

\[
\Delta \mathbf{G}(r, r', \omega) = \left( \frac{\omega}{c} \right)^2 \int d^3 s \chi_e(s, \omega) \mathbf{G}(r, s, \omega) \mathbf{G}(s, r', \omega).
\]

(7)

According to Eq. (1), the resulting change of the CP potential is

\[
\Delta U_A(r_A) = -\frac{\hbar}{2\pi\varepsilon_0} \int_0^\infty d u \left( \frac{u}{c} \right)^4 \alpha_A(iu) \times \int d^3 s \chi_e(s, iu) \text{Tr} [ \mathbf{G}(r_A, s, iu) \mathbf{G}(s, r_A, iu) ].
\]

(8)

Recalling Eq. (6), one can easily see that \( \Delta U_A \) is just an integral over two-atom potentials \( U_{AB}(r_A, r_B) \),

\[
\Delta U_A(r_A) = \int_{V_b} d^3 r_B \eta_b(r_B) U_{AB}(r_A, r_B). \]

(9)
where

\[ U_{AB}(\mathbf{r}_A, \mathbf{r}_B) = -\frac{\hbar}{2\pi \varepsilon_0} \int_0^\infty \frac{du}{u^2} \alpha_A(iu) \alpha_B(iu) \]

(10)

\[ \times \text{Tr}\{ \mathbf{G}(\mathbf{r}_A, \mathbf{r}_B, iu) \mathbf{G}(\mathbf{r}_B, \mathbf{r}_A, iu) \} \]

is the vdW potential between two polarizable atoms in the presence of an arbitrary arrangement of dispersing and absorbing magnetoelectric bodies. The total force acting on atom A due to atom B(A) and the bodies is just the sum of the single-atom CP force (2) and the two-atom vdW force

\[ \mathbf{F}_{AB(BA)} = -\nabla \chi_{AB}(\mathbf{r}_A, \mathbf{r}_B), \]

(11)

where in general, \( \mathbf{F}_{AB} \neq -\mathbf{F}_{BA} \) due to the presence of the bodies. Equation (10) agrees with the result obtained on the basis of fourth-order perturbation theory, with the perturbative calculation being much more lengthy [29, 30]. Needless to say, the method presented here can be easily extended to derive \( N \)-atom potentials [29, 31], whereas the perturbative method becomes increasingly cumbersome for large \( N \).

The vdW potential between a polarizable atom A and a magnetizable atom B can be derived in a highly analogous way. Again, we start from atom A placed within an arbitrary arrangement of magnetoelectric bodies, Eqs. (1) and (4), but this time we add a weakly magnetizable body consisting of a collection of magnetizable atoms of type B. For sufficiently small atomic number densities, the magnetic susceptibility of this body approximately reads

\[ \chi_m(\mathbf{r}, \omega) = \begin{cases} \mu_0 \eta_B(\mathbf{r}) \beta_B(\omega) & \text{if } \mathbf{r} \in V_b \\ 0 & \text{if } \mathbf{r} \not\in V_b \end{cases} \]

(12)

with \( \beta_B(\omega) \) denoting the magnetizability. The inverse permeability of the total arrangement of bodies reads \( \kappa(\mathbf{r}, \omega) = \chi_m(\mathbf{r}, \omega) \), so that the Green tensor corresponding to this arrangement is given by Eq. (4) with \( \kappa(\mathbf{r}, \omega) = \chi_m(\mathbf{r}, \omega) \) instead of \( \kappa(\mathbf{r}, \omega) \). A linear expansion in terms of \( \chi_m(\mathbf{r}, \omega) \) leads to

\[ \Delta \mathbf{G}(\mathbf{r}, \mathbf{r'}, \omega) = -\int d^3 s \chi_m(\mathbf{s}, \omega) \left[ \mathbf{G}(\mathbf{r}, \mathbf{s}, \omega) \times \nabla_s \right] \nabla_s \times \mathbf{G}(\mathbf{s}, \mathbf{r'}, \omega). \]

(13)

Combining this with Eq. (1), we find that the change in the CP potential due to the presence of the additional magnetizable body is

\[ \Delta U_A(\mathbf{r}_A) = -\frac{\hbar}{2\pi \varepsilon_0} \int_0^\infty \frac{du}{u^2} \left[ \alpha_A(iu) \int d^3 s \chi_m(\mathbf{s}, iu) \right] \]

\[ \times \text{Tr}\{ \mathbf{G}(\mathbf{r}_A, \mathbf{s}, iu) \times \nabla_s \nabla_s \times \mathbf{G}(\mathbf{s}, \mathbf{r}_A, iu) \} \]

(14)

Finally, upon using Eq. (12), Eq. (14) can be rewritten in the form of Eq. (9), where now

\[ U_{AB}(\mathbf{r}_A, \mathbf{r}_B) = -\frac{\hbar}{2\pi \varepsilon_0} \int_0^\infty \frac{du}{u^2} \alpha_A(iu) \beta_B(iu) \]

(15)

\[ \times \text{Tr}\{ \mathbf{G}(\mathbf{r}_A, \mathbf{r}, iu) \times \nabla_r \nabla_r \times \mathbf{G}(\mathbf{r}, \mathbf{r}_A, iu) \} \] \( r = r_a \)

is the vdW potential between a polarizable atom A and a magnetizable atom B in the presence of an arbitrary arrangement of magnetoelectric bodies. To our knowledge, Eq. (15) has never been derived before.

3. EXAMPLES

3.1. Free Space

In order to illustrate the two-atom vdW potentials (10) and (15), let us first consider two atoms in free space, with the Green tensor being given by [32]

\[ \mathbf{G}^{(0)}(\mathbf{r}, \mathbf{r'}, \omega) = \left\{ \begin{array}{ll} \frac{\hbar}{4\pi \rho} \{ a[c/(up)]I - b[c/(up)]e_s e_p \}, & \rho = |\mathbf{r} - \mathbf{r'}| < |\mathbf{r}| \leq |\mathbf{r'}|; \\ 0, & |\mathbf{r}| > |\mathbf{r'}|. \end{array} \right. \]

\[ a(x) = 1 + x + x^2, \quad b(x) = 1 + 3x + 3x^2 \]

(17)

\[ (\mathbf{p} = \mathbf{r} - \mathbf{r'}; \mathbf{p} = |\mathbf{p}|; e_s = \mathbf{p}/|\mathbf{p}|; I, \text{ unit tensor}). \] In this case, the vdW potential between two polarizable atoms, Eq. (10), reads

\[ U_{AB}^{(0)}(\mathbf{r}_A, \mathbf{r}_B) = -\frac{\hbar}{32\pi^3} \int_0^\infty \frac{du}{u^2} \alpha_A(iu) \beta_B(iu) g(ull/c), \]

(18)

\[ g(x) = 2e^{-2x} \left( 3 + 6x + 5x^2 + 2x^3 + x^4 \right) \]

(19)

\( l = |\mathbf{r}_A - \mathbf{r}_B| \), in agreement with the well-known result of Casimir and Polder [2]. From

\[ \nabla \left[ g(ull/c) \right] = -\frac{4\pi}{r^2} \left[ e^{-2x} (9 + 18x + 16x^2 + 8x^3 + 3x^4 + x^5) \right] |_{x = ull/c}, \]

(20)

it can be seen that potential (18) is always attractive.

In the retarded limit, where \( l \gg c/\omega_{\min} \) (\( \omega_{\min} \) denoting the minimum of all resonance frequencies of atoms A and B), the exponential factor effectively limits the \( u \)-integral in Eq. (18) to a region where

\[ \alpha_A(iu) = \alpha_A(0), \quad \alpha_B(iu) = \alpha_B(0), \]

(21)

hence, the integral can be performed in closed form to yield

\[ U_{AB}^{(0)}(\mathbf{r}_A, \mathbf{r}_B) = -\frac{C}{l}, \]

(22)
In the opposite, nonretarded limit, where \( l \ll c/\omega_{\text{max}} \) (\( \omega_{\text{max}} \) denoting the maximum of all resonance frequencies of atoms \( A \) and \( B \)), the factors \( \alpha_A(iu) \) and \( \alpha_B(iu) \) effectively limit the \( u \)-integral in Eq. (18) to a region where

\[
g(u/l) = g(0) = 6,
\]

so that the London potential [1] is recovered,

\[
U_{AB}^{(0)}(r_A, r_B) = -\frac{C_6}{l^6},
\]

\[
C_6 = \frac{3\hbar}{16\pi^2} \int_0^\infty du \alpha_A(iu)\alpha_B(iu).
\]

To calculate the potential between a polarizable atom and a magnetizable one, we first use Eq. (16) to derive

\[
\nabla \times \mathbf{G}^{(0)}(r, r', iu) = \frac{e^{-u/l}}{(4\pi \rho)} \left[ (1 + \frac{\rho}{c}) \mathbf{e}_\rho \times \mathbf{I} \right],
\]

\[
\mathbf{G}^{(0)}(r, r', iu) \times \nabla = \frac{e^{-u/l}}{(4\pi \rho)} \left[ (1 + \frac{\rho}{c}) \mathbf{I} \times \mathbf{e}_\rho \right].
\]

Substituting Eqs. (27) and (28) into Eq. (15) and using the identity

\[
\text{Tr}[\mathbf{e}_\rho \times \mathbf{I} \times \mathbf{e}_\rho] = -2,
\]

we find that

\[
U_{AB}^{(0)}(r_A, r_B) = \frac{\hbar \mu_0}{2\pi^2} \int_0^\infty du \frac{\alpha_A(iu)^2 \beta_B(iu)h(u/l)}{I^4},
\]

\[
h(x) = 2e^{-2x}(1 + 2x + x^2)
\]

in agreement with results found earlier [24, 25].

In contrast to the attractive vdW potential between two polarizable atoms, Eq. (18), the vdW potential between a polarizable and a magnetizable atom, Eq. (30), is always repulsive, as can be seen from

\[
\nabla \left[ \frac{h(u/l)}{I^4} \right] = -\frac{4\pi}{I^2} \left[ e^{-2x}(2 + 4x + 3x^2 + x^3) \right]_{x = u/l}.
\]

In particular, one finds, upon using Eqs. (21) and (33), that Eq. (30) reduces to

\[
U_{AB}^{(0)}(r_A, r_B) = \frac{C_7}{l^7},
\]

\[
C_7 = \frac{23\hbar c \alpha_A(0)\alpha_B(0)}{64\pi^3 \varepsilon_0^2}.
\]
\[ G^{(1)}_{zz}(r_A, r_B, iu) = \frac{1}{4\pi} \int_0^\infty dq q e^{-b_\perp Z J_1(qX)}/k^2 r_p, \quad (39) \]

\[ G^{(1)}_{z}(r_A, r_B, iu) = -\frac{1}{4\pi} \int_0^\infty dq q e^{-b_\perp Z J_0(qX)}/bk^2 r_p \quad (40) \]

\[ [J_\perp(x); \text{Bessel function; } Z = z_A + z_B; X = x_B - x_A], \text{where } r_\perp = r_\perp(q, u) \text{ (} \sigma = s, p \text{)} \text{ are the reflection coefficients of the half space } \text{(cf. Eqs. (46), (63), and (64) below), and} \]

\[ b = b(q, u) = \frac{u^2}{c^2} + q^2, \quad (41) \]

\[ k = k(q, u) = u/c. \quad (42) \]

According to the decomposition (3) of the Green tensor, the two-atom potential (10) can be split into three parts,

\[ U_{AB}(r_A, r_B) = U^{(0)}_{AB}(r_A, r_B) + U^{(1)}_{AB}(r_A, r_B) + U^{(2)}_{AB}(r_A, r_B), \quad (43) \]

where the bulk-part contribution \( U^{(0)}_{AB}(r_A, r_B) \) is simply the free-space result (18),

\[ U^{(1)}_{AB}(r_A, r_B) = -\frac{\hbar}{\pi^2} \int_0^\infty \left[ \alpha_A((iu)\alpha_B((iu)) \times \text{Tr}[G^{(0)}(r_A, r_B, iu)G^{(1)}(r_B, r_A, iu)] \right. \]

\[ \left. \times e^{-b_\perp Z J_1(qX)}/k^2 r_p \right] \quad (44) \]

\[ \text{comes from the cross term of bulk and scattering parts (with } Z = z_B - z_A, \xi = c/(lu), a(x) \text{ and } b(x) \text{ are as given in Eq. (17)), and} \]

\[ U^{(2)}_{AB}(r_A, r_B) = -\frac{\hbar}{2\pi^2} \int_0^\infty \left[ \alpha_A((iu)\alpha_B((iu)) \times \text{Tr}[G^{(1)}(r_A, r_B, iu)G^{(1)}(r_B, r_A, iu)] \right. \]

\[ \left. \times \text{Tr}[G^{(1)}(r_A, r_B, iu)G^{(1)}(r_B, r_A, iu)] \right] \quad (45) \]

is the scattering-part contribution \( (b' = b(q', u), r_\sigma = r_\sigma(q', u) \text{ for } \sigma = s, p) \).

### 3.2.1. Perfectly reflecting plate

For a perfectly reflecting plate, the reflection coefficients are simply given by

\[ r_z = \mp 1, \quad r_p = \pm 1, \quad (46) \]

where the upper (lower) sign corresponds to a perfectly conducting (permeable) plate. In the retarded limit, where \( l, z_A, z_B \gg c/\omega_{\text{min}}, U^{(0)}_{AB} \text{ is given by Eq. (22), whereas } U^{(1)}_{AB} \text{ (Eq. (44)) and } U^{(2)}_{AB} \text{ (Eq. (45)) can be given in closed form only in some special cases. If } X \ll Z_+ \text{ (cf. Fig. 1), we derive, on using the relevant elements of the scattering Green tensor as given in Appendix A (Eqs. (A10) and (A11),}

\[ U^{(1)}_{AB} = \pm \frac{32}{23} \frac{(X^2 + 6l^2)C_7}{l^3} \quad \text{and} \quad (47) \]

\[ U^{(2)}_{AB} = \frac{C_7}{Z_+}, \quad (48) \]

where \( C_7 \) is given by Eq. (23). Thus, recalling Eq. (22), the two-atom vdw potential (43) reads

\[ U_{AB} = \frac{C_7}{l^3} \pm \frac{32}{23} \frac{(X^2 + 6l^2)C_7 - C_7}{l^3} \quad \text{and} \quad (49) \]

In particular, if \( z_A/z_B \ll 1, \text{ Eqs. (47) and (48) imply that} \]

\[ U^{(1)}_{AB} = \pm \frac{6}{23} U^{(0)}_{AB}, \quad (50) \]

\[ U^{(2)}_{AB} = U^{(0)}_{AB}, \quad (51) \]
so the presence of the perfectly reflecting plate leads to an enhancement of the interaction potential,

$$U_{AB} = \begin{cases} 
\frac{40}{23} U_{AB}^{(0)} \\
\frac{52}{23} U_{AB}^{(0)}
\end{cases}$$

(52)

for a perfectly conducting or permeable plate, respectively.

Quite generally, since the bulk part $U_{AB}^{(0)}$ (first term on the r.h.s. of Eq. (49)) is negative, the interaction potential is enhanced (reduced) by the plate if the scattering part $U_{AB}^{(1)} + U_{AB}^{(2)}$ (second and third terms on the r.h.s. of Eq. (49)) is negative (positive). In the case of a perfectly conducting plate, it is seen that especially for $Z = 0$, briefly referred to as the parallel case, $U_{AB}^{(1)} + U_{AB}^{(2)}$ is positive and, hence, the interaction potential is reduced by the plate, whereas for $X = 0$, briefly referred to as the vertical case, $U_{AB}^{(1)} + U_{AB}^{(2)}$ is positive and the interaction potential is reduced iff

$$z_B/z_A \approx 4.90,$$

(53)

where, without a loss of generality, atom A is assumed to be closer to the plate than atom B. It is apparent from Eq. (49) that for a perfectly permeable plate, $U_{AB}^{(1)} + U_{AB}^{(2)}$ is always negative and, hence, the interaction potential is always enhanced by the plate.

In the nonretarded limit, where $l$, $z_A$, $z_B \ll c/\omega_{\max}$, $U_{AB}^{(0)}$ is given by Eq. (25), and from Eqs. (44) and (45) we derive, on making use of the relevant elements of the scattering Green tensor as given in Appendix (Eqs. (A14)–(A17)),

$$U_{AB}^{(1)} = \pm \frac{[4X^4 - 2X^2Z^2 + X^2(Z^2_+ + Z^2_-)]C_6}{3l^5l_+^5},$$

(54)

$$U_{AB}^{(2)} = \frac{C_6}{l_+^6}$$

(55)

($l_+ = \sqrt{X^2 + Z^2}$), where $C_6$ is given by Eq. (26). Hence, the interaction potential (43), reads, on recalling Eq. (25),

$$U_{AB} = \frac{-C_6}{l_+^6}$$

$$\pm \frac{[4X^4 - 2X^2Z^2 + X^2(Z^2_+ + Z^2_-)]C_6}{3l^5l_+^5} - \frac{C_6}{l_+^6},$$

(56)

Sign of $U_{AB}^{(1)}$ for a perfectly reflecting plate

<table>
<thead>
<tr>
<th>Case</th>
<th>Conducting plate</th>
<th>Permeable plate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallel case</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>Vertical case</td>
<td>−</td>
<td>+</td>
</tr>
</tbody>
</table>

Let us again consider the effect of the plate on the interaction potential for the parallel and vertical cases. In the parallel case, Eq. (56) takes the form

$$U_{AB} = \frac{-C_6}{l_+^6} \pm \frac{(4l^2 + Z_+^2)C_6}{3l^3l_+^5} - \frac{C_6}{l_+^6},$$

(57)

which, in the on-surface limit $Z_+ \longrightarrow 0$, approaches

$$U_{AB} = \begin{cases} 
\frac{2}{3} U_{AB}^{(0)} \\
\frac{10}{3} U_{AB}^{(0)}
\end{cases}$$

(58)

for a perfectly conducting or permeable plate, respectively. It can easily be seen that the term $U_{AB}^{(1)}$ (second term on the r.h.s. of Eq. (57)) dominates the term $U_{AB}^{(2)}$ (third term on the r.h.s. of Eq. (57)), so $U_{AB}^{(1)} + U_{AB}^{(2)}$ is positive (negative) for a perfectly conducting (permeable) plate and, hence, the interaction potential is reduced (enhanced) due to the presence of the plate.

In the vertical case, from Eq. (56), the interaction potential is obtained to be

$$U_{AB} = \frac{-C_6}{l_+^6} - \frac{2C_6}{3Z_+ l_+^3} - \frac{C_6}{Z_+},$$

(59)

It is obvious that $U_{AB}^{(1)} + U_{AB}^{(2)}$ (second and third terms on the r.h.s. of Eq. (59)) is negative when the plate is perfectly conducting, thereby enhancing the interaction potential since $U_{AB}^{(0)}$ (first term in Eq. (59)) is negative. In the case of a perfectly permeable plate, $U_{AB}^{(1)} + U_{AB}^{(2)}$ is positive iff

$$\frac{z_B}{z_A} < 1 + \frac{2}{(3/2)^{1/3}} = 14.82,$$

(60)

where atom A is again assumed to be closer to the plate than atom B.

The enhancing/reducing effect of the perfectly reflecting plate on the two-atom vdW potential in the various cases considered can be systematized in a simple way. Since $U_{AB}^{(0)}$ and $U_{AB}^{(2)}$ are negative in all cases, the enhancement or reduction of the vdW potential due to the presence of the plate depends only on the sign of $U_{AB}^{(1)}$ and its magnitude compared to that of $U_{AB}^{(2)}$. Moreover, the results for the nonretarded limit (the sign
of $U_{AB}^{(1)}$ being summarized in table) can be explained by using the method of image charges, where the two-atom vdW interaction is regarded as being due to the interactions between fluctuating dipoles $A$ and $B$ and their images $A'$ and $B'$ in the plate, with

$$\hat{H}_{\text{int}} = \hat{V}_{AB} + \hat{V}_{AB}' + \hat{V}_{BA}'$$

being the corresponding interaction Hamiltonian. Here, $\hat{V}_{AB}$ denotes the direct interaction between dipole $A$ and dipole $B$, while $\hat{V}_{AB}'$ and $\hat{V}_{BA}'$ denote the indirect interaction between each dipole and the image induced by the other one in the plate. According to this approach, the vdW potential $U_{AB}$ can be identified by the second-order energy shift

$$\Delta E_{AB} = - \sum_{(n, m) \neq (0, 0)} \frac{\langle 0_A | \langle 0_B | \hat{H}_{\text{int}} | n_A \rangle | m_B \rangle}{\hbar} \left( E_A^n + E_B^m - E_A^0 - E_B^0 \right)$$

$$\times \langle n_A | \langle m_B | \hat{H}_{\text{int}} | 0_A \rangle | 0_B \rangle$$

(E$_{A(B)}^n$, $|n_{A(B)}\rangle$, atomic eigenenergies and eigenstates, respectively). In this approach, $U_{AB}^{(0)}$ corresponds to the product of two direct interactions, so it is negative in agreement with Eq. (56) because of the minus sign on the r.h.s. of Eq. (62). Accordingly, $U_{AB}^{(2)}$ is due to the product of two indirect interactions and is also negative in agreement with Eq. (56). The terms containing one direct and one indirect interaction are contained in $U_{AB}^{(1)}$ and determine its sign. We can thus predict the sign of $U_{AB}^{(1)}$ from a graphical construction of the image charges, as sketched in Figs. 2--5.

Figure 2 shows two electric dipoles in front of a perfectly conducting plate in the parallel case. The configuration of dipoles and images indicates repulsion between dipole $A$ and dipole $B$, so $U_{AB}^{(1)}$ is positive, in agreement with the table. On the contrary, in the vertical case from Fig. 3, attraction is indicated, i.e., negative $U_{AB}^{(1)}$, which is also in agreement with the table.

The case of two electric dipoles in front of a perfectly permeable plate can be treated by considering two magnetic dipoles in front of a perfectly conducting plate, as the two situations are equivalent due to the duality between electric and magnetic fields in the absence of free charges or currents. From Figs. 4 (parallel case) and 5 (vertical case), it is apparent that the interaction between dipole $A(B)$ and dipole $B'(A')$ is
attractive in the parallel case and repulsive in the vertical case, again confirming the sign of \( U_{AB}^{(1)} \) as given in the table.

When the dipole–dipole separation in Fig. 5 is sufficiently small compared with the dipole–surface separations, the direct interaction between the two dipoles is expected to be stronger than their indirect interaction via the image dipoles. As a result, \( U_{AB}^{(1)} \) will be the dominant term in \( U_{AB}^{(1)} + U_{AB}^{(2)} \) and \( U_{AB}^{(1)} + U_{AB}^{(2)} \) becomes positive. However, when the dipole–dipole separation exceeds the dipole–surface separations, then the indirect interaction may become comparable to the direct one and \( U_{AB}^{(2)} \) may be the dominant term, leading to negative \( U_{AB}^{(1)} + U_{AB}^{(2)} \). The image dipole model thus also gives a qualitative explanation of the condition (60).

### 3.2.2. Magnetolectric half space

Let us now abandon the assumption of perfect reflectivity and consider a magnetolectric half space of permittivity \( \varepsilon(0) \) and permeability \( \mu(0) \). In this case, the reflection coefficients in Eqs. (44) and (45) are given by

\[
\begin{align*}
r_s &= r_s(q, u) = \frac{\mu(iu)b - b_M}{\mu(iu)b + b_M}, \quad (63) \\
r_p &= r_p(q, u) = \frac{\varepsilon(iu)b - b_M}{\varepsilon(iu)b + b_M}, \quad (64)
\end{align*}
\]

where \( b \) is defined by Eq. (41); and

\[
b_M = b_M(q, u) = \sqrt{\frac{\varepsilon(iu)\mu(iu)u^2}{c^2} + q^2}. \quad (65)
\]

In the retarded limit, \( l, z_A, z_B \gg c/\omega_{\text{min}} \) (where \( \omega_{\text{min}} \) now denotes the minimum of all resonance frequencies of atoms \( A \) and \( B \) and the magnetoatomic medium), we may again approximate the atomic polarizabilities by their static values, recall Eq. (21), and, similarly, we may set

\[
\varepsilon(iu) = \varepsilon(0), \quad \mu(iu) = \mu(0). \quad (66)
\]

Replacing the integration variable \( q \) in Eq. (44) by \( \nu = bc\nu/u \) (cf. Eq. (A9)), one can show that the contribution \( U_{AB}^{(1)} \) to the vdW potential takes the form

\[
U_{AB}^{(1)}(r_A, r_B) = \frac{hc}{32\pi^3 \varepsilon_0^2} \alpha_A(0) \alpha_B(0) + l^2 A_{s+} + lA_{s+} + A_{s+},
\]

\[
+ 2(\nu^2 - 1) \left[ X^2 B_3 + (X^2 - 2Z^2) \left( \frac{B_1}{l} + \frac{B_3}{l^2} \right) \right] r_p
\]

\[
+ \left[ Z^2 A_{s+} + (Z^2 - 2X^2) \left( \frac{A_{s+}}{l} + \frac{A_{3+}}{l^2} \right) \right] r_s
\]

where according to Eqs. (63) and (64), the static reflection coefficients are given by

\[
r_s = r_s(\nu) = \frac{\mu(0)\nu - \sqrt{\varepsilon(0)\mu(0) - 1 + \nu^2}}{\mu(0)\nu + \sqrt{\varepsilon(0)\mu(0) - 1 + \nu^2}}, \quad (68)
\]

\[
r_p = r_p(\nu) = \frac{\varepsilon(0)\nu - \sqrt{\varepsilon(0)\mu(0) - 1 + \nu^2}}{\varepsilon(0)\nu + \sqrt{\varepsilon(0)\mu(0) - 1 + \nu^2}}, \quad (69)
\]

and

\[
A_{k\pm} = \int dxxx e^{i\kappa x}[J_0(\lambda x) \pm J_2(\lambda x)], \quad (70)
\]

\[
B_k = \int dxxx e^{-i\kappa x} J_0(\lambda x), \quad (71)
\]

with \( \lambda = l + \nu Z_x \) and \( \zeta = X\sqrt{\nu^2 - 1} \) (for explicit expressions of \( A_{k\pm} \) and \( B_k \), see Appendix B). Similarly, Eq. (45) reduces to

\[
U_{AB}^{(2)} = -\frac{hc}{64\pi^3 \varepsilon_0^2} \alpha_A(0) \alpha_B(0)
\]

\[
\times \int d\nu \int d\nu' \{(r_p r_p' [3 \nu^2 \nu'^2 - 2(\nu^2 + \nu'^2)] + 2]
\]

\[
+ r_s r_s' - r_s' r_s \nu^2 - r_p r_p' \nu^2 \} M_0
\]

\[
+ 4\nu\nu' \sqrt{\nu^2 - 1} \sqrt{\nu'^2 - 1} r_p r_p' M_1
\]

\[
+ (r_s r_s' + r_p r_p' \nu^2 + r_s' r_p' \nu'^2 + r_p r_p' \nu^2) M_2
\]

\[
(r_p' = r_p(\nu') \text{ for } \sigma = s, p), \quad (72)
\]

where

\[
M_\nu = \int dxx e^{-(\nu + \nu')Z_x} J_0(\zeta x) J_0(\zeta' x), \quad (73)
\]
Retarded limit, of atoms tensor being as given in Appendix A. In the case of a
\[ \omega = \text{constant}, \] then, approximately,
\[ \left( 4X^4 - 2Z^2Z^2_+ + X^2(Z^2 + Z^2_+) \right) \frac{D}{l^6} - \frac{E}{l^6}, \] (75)
where \( C_6 \) is given by Eq. (26), and
\[ D = \frac{\hbar}{16\pi^3\varepsilon_0^2} \int \text{d}u \alpha_A(iu) \alpha_B(iu) \left( \frac{\varepsilon(iu) - 1}{\varepsilon(iu) + 1} \right)^2, \] (76)
\[ E = \frac{3\hbar}{16\pi^3\varepsilon_0^2} \int \text{d}u \alpha_A(iu) \alpha_B(iu) \left( \frac{\varepsilon(iu) - 1}{\varepsilon(iu) + 1} \right)^2. \] (77)

In particular, in the limiting case when \( l \ll Z_+ \), Eq. (75) reduces to
\[ U_{AB} = -\frac{C_6}{l^6} + \frac{(X^2 - 2Z^2)D}{l^5 Z^+_+}, \] (78)

It is seen that the second term on the r.h.s. of this equation is positive (negative) in the parallel (vertical) case, so the vdW potential is reduced (enhanced) by the presence of the electric half space.

In the case of a purely magnetic half space (\( \varepsilon = 1 \)), we derive (Eqs. (A24)–(A27))
\[ U_{AB} = -\frac{C_6}{l^6} + \frac{Z^2 + 3Z_+(l_+ - Z_+)}{l^5 Z^+_+ F}, \] (79)
where
\[ F = \frac{\hbar}{64\pi^3\varepsilon_0^2} \int \text{d}u \left( \frac{\mu}{c} \right)^2 \alpha_A(iu) \alpha_B(iu) \times \left( \frac{\mu(iu) - 1}{\mu(iu) + 1} \right)^3. \] (80)

Note that \( U_{AB}^{(2)} \) does not contribute to the asymptotic nonretarded vdW potential \( U_{AB} \) for the purely magnetic half space. In particular, in the limiting case when \( X \ll Z_+ \), Eq. (79) reduces to
\[ U_{AB} = -\frac{C_6}{l^6} + \frac{(2Z^2 - X^2)F}{2l^5 Z^+_+}. \] (81)

It is seen that the second term in the r.h.s. of this equation is negative (positive) in the parallel (vertical) case, so the vdW potential is enhanced (reduced) due to the presence of the magnetic half space.

It should be pointed out that the nonretarded limit for the magnetoelectric half space is in general incom-
compatible with the limit of perfect reflectivity \( \varepsilon(iu) \to \infty \) or \( \mu(iu) \to \infty \) considered in Section 3 B1, as is clearly seen from the condition given above Eq. (75) (cf. also the expansions (A18) and (A19), which are not well-behaved in the limit of perfect reflectivity). As a consequence, Eq. (79) does not reduce to Eq. (56) via the limit \( \mu(iu) \to \infty \). It is therefore remarkable that the result for a purely electric half space, Eq. (75), reduces to Eq. (56) in the limit \( \varepsilon(iu) \to \infty \), as already noted in [33] in the case of the single-atom potential.

Figures 6–8 show the results of an exact (numerical) calculation of the vdW interaction between two identical atoms near a semi-infinite half space, as given by Eq. (43) together with Eqs. (18), (44), and (45), as well as Eqs. (63) and (64). In the calculations, we have used single-resonance models for both the polarizability of the atoms,

\[
\alpha_A(\omega) = \alpha_B(\omega) = \lim_{\varepsilon \to \infty} \frac{2}{\hbar} \frac{\omega_{10} |d_{10}|^2}{\omega_{10}^2 - \omega^2 - i\omega \varepsilon} \tag{82}
\]

(with \( \omega_{10} \) and \( d_{10} \) denoting the frequency and electric dipole matrix element of the dominant atomic transition, respectively), and the permittivity and permeability of the half space,

\[
\varepsilon(\omega) = 1 + \frac{\omega_{pe}^2}{\omega_{Fe}^2 - \omega^2 - i\omega \gamma_e}, \tag{83}
\]

\[
\mu(\omega) = 1 + \frac{\omega_{pm}^2}{\omega_{Fm}^2 - \omega^2 - i\omega \gamma_m}. \tag{84}
\]
In the figures, the potentials and the forces are normalized w.r.t. their values in free space as given by Eq. (18), so one can clearly see that the vdW interaction is unaffected by the presence of the half space for atom-half space separations that are much greater than the interatomic separations (the curves approaching unity for \( z_A, z_B \gg l \)), while an asymptotic enhancement or reduction of the interaction is observed in the opposite limit.

Figure 6a shows the dependence of the normalized vdW potential on the atom-atom separation \( l \) in the parallel case \((Z = 0)\) for different values of the distance \( z_A = z_B \) of the atoms from a purely electric half space. The ratio of the interatomic force along the connecting line of the two atoms, \( F_{AB}^{(0)}(l) \) [Eq. (11)], to the corresponding force in free space, \( F_{AB}^{(0)}(l) \), closely follows the ratio \( U_{AB}(l)/U_{AB}^{(0)}(l) \), so that within the resolution of the figures, the curves for \( F_{AB}(l)/F_{AB}^{(0)}(l) \) (not shown) would almost coincide with those for \( U_{AB}(l)/U_{AB}^{(0)}(l) \). The figure reveals that due to the presence of the electric half space, the attractive vdW potential and force are reduced, in agreement with the predictions from the nonretarded limit, Eq. (78). The relative reductions of the potential and the force are not monotonic; there is a value of the atom–atom separation where the reduction is strongest.

The \( l \)-dependence of \( U_{AB}(l)/U_{AB}^{(0)}(l) \) in the presence of a purely magnetic half space in the parallel case is shown in Fig. 6b. The corresponding force ratio \( F_{AB}(l)/F_{AB}^{(0)}(l) \) (not shown) again behaves like \( U_{AB}(l)/U_{AB}^{(0)}(l) \). The figure indicates that the presence of a purely magnetic half space enhances the vdW interaction between the two atoms, with the enhancement increasing with the atom-atom separation, in agreement with the nonretarded limit, Eq. (81).

Figure 7 shows \( U_{AB}(l)/U_{AB}^{(0)}(l) \) in the vertical case \((X = 0)\) when the half space is purely electric (Fig. 7a) or purely magnetic (Fig. 7b). In the figures, atom \( A \) is assumed to be closer to the surface of the half space than atom \( B \) and the graphs show the variation of the vdW potential with the atom–atom separation \( l \) for different distances \( z_A \) of atom \( A \) from the half space. It is seen that for a purely electric half space, the potential is enhanced compared to the one observed in the free-space case, in agreement with Eq. (78). Note that there are values of the atom–atom separation at which the enhancement is strongest.

For a purely magnetic half space, the potential is seen to be typically enhanced, although for very small atom–atom separations, a reduction appears (inset in Fig. 7b), in agreement with Eq. (81). Due to this slight reduction for small atom–atom separations, the relative enhancement is not monotonous, in contrast to what is suggested by the large figure.

Whereas the force \( F_{BA}(l)/F_{BA}^{(0)}(l) \) for the force acting on atom \( B \) (not shown) again follows closely the potential ratio \( U_{AB}(l)/U_{AB}^{(0)}(l) \) for both electric and magnetic half spaces (as in Fig. 7, not shown), the ratio \( F_{BA}(l)/F_{BA}^{(0)}(l) \), for the force acting on atom \( A \) differs noticeably from \( U_{AB}(l)/U_{AB}^{(0)}(l) \) (Fig. 8). Clearly, the difference is due to the fact that the atom \( A(B) \), which is responsible for the force \( F_{BA}(l) \), is situated on the same side of atom \( B(A) \) as the half space in the former case, but on a different side in the latter case (cf. Figs. 3 and 5).

4. SUMMARY AND CONCLUSIONS

Starting from the CP potential of a single polarizable atom placed within a given arrangement of magnetoelectric bodies, we have presented a macroscopic derivation of two-atom vdW potentials. By introducing an additional weakly polarizable body and linearly expanding the resulting CP interaction in terms of the body’s susceptibility, the vdW potential between two polarizable atoms in the presence of an arbitrary arrangement of dispersing and absorbing magnetoelectric bodies has been inferred. The vdW potential between a polarizable atom and a magnetizable one has been derived in a similar way by introducing a weakly magnetizable body. The general formulas have been used to study the influence of polarizability and magnetizability on the vdW potential between two atoms in free space. In particular, it has been shown that the vdW interaction of a polarizable atom with a magnetizable one is always repulsive, in contrast to the well-known attractive potential between two polarizable atoms.

To illustrate the influence of the presence of magnetoelectric bodies on the vdW potential, we have considered the example of two polarizable atoms near a perfectly reflecting plate. It has turned out that due to the presence of the plate, the attractive vdW interaction between the atoms can be enhanced or reduced depending on the magnetoelectric properties of the plate and the specific alignment of the atoms with respect to the plate. In particular, in the nonretarded limit, these effects can be qualitatively explained using the method of image dipoles. To be more realistic, we have also calculated the vdW potential for the case of the two atoms near a magnetoelectric half space of finite permittivity and permeability. The analytical results show that in the nonretarded limit, the potential in the case of a purely electric half space is reduced (enhanced) compared to its value in free space in the case of parallel (vertical) alignment of the two atoms, while in the case of a purely magnetic half space, it is enhanced (reduced) for the parallel (vertical) alignment of the two atoms. The numerical computation of the potential in the whole
distance regime confirms the analytical results. In addition, it shows that the relative enhancement/reduction of the \(v\)d/W interaction is not always monotonous, but may in general display maxima or minima, in particular in the case of a purely electric half space.

In conclusion, the examples studied in this work suggest that the sign of the \(v\)d/W potential is entirely determined by the electric/magnetic nature of the interacting atoms, while the strength of the respective attractive or repulsive potentials can be controlled by the presence of magnetoelectric bodies.

**APPENDIX A**

Scattering Green Tensor for a Semi-Infinite Half Space

The scattering Green tensor for a semi-infinite half space can be given in the form [34]

\[
G^{(1)}(r, r', iu) = \int d^2 q e^{i q(r-r')} G^{(1)}(q, z, z', iu)
\]  

(A1)

\(q \perp \mathbf{e}_z\), where

\[
G^{(1)}(q, z, z', iu) = \frac{1}{8 \pi^2 b \sigma} \sum_{s, p} e_s^e e_p^e e^{i(\mathbf{z}+\mathbf{z}')}
\]

(A2)

with

\[
e_s^e = \sin \phi \mathbf{e}_x - \cos \phi \mathbf{e}_y,
\]

\[
e_p^e = \pm \frac{b}{k} (\cos \phi \mathbf{e}_x + \sin \phi \mathbf{e}_y) - \frac{i q}{k} \mathbf{e}_z
\]

(A3)

(A4)

\(e_q = \cos \phi \mathbf{e}_x + \sin \phi \mathbf{e}_y = q/q, q = |q|\) denoting the polarization vectors for s- and p-polarized waves propagating in the positive(+)negative(−) \(z\)-direction. Further, \(b\) and \(k\) are defined according to Eqs. (41) and (42), respectively, and the reflection coefficients \(r_s, r_p\) are given by Eq. (46) for a perfectly reflecting plate and by Eqs. (63) and (64) for a magnetoelectric half space. Equations (A3) and (A4) imply that

\[
e_s^e e_s^e = \begin{pmatrix}
\sin^2 \phi & -\sin \phi \cos \phi & 0 \\
-\sin \phi \cos \phi & \cos^2 \phi & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

(A5)

\[
e_p^e e_p^e = \begin{pmatrix}
\frac{b^2}{k^2} \cos^2 \phi & \frac{b^2}{k^2} \sin \phi \cos \phi \frac{ib}{k^2} \cos \phi \\
\frac{b^2}{k^2} \sin \phi \cos \phi & \frac{b^2}{k^2} \sin^2 \phi \frac{ib}{k^2} \sin \phi \\
\frac{ib}{k^2} \cos \phi & \frac{ib}{k^2} \sin \phi & \frac{-q^2}{k^2}
\end{pmatrix}
\]

(A6)

Substituting these results into Eqs. (A1) and (A2), performing the \(\phi\)-integrals by means of [35]

\[
\int d\phi e^{i q \cos \phi} \cos (vx) = 2\pi i J_0(x)
\]

(A7)

and using the relation

\[
J_1(x) = \frac{J_0(x) - J_2(x)}{2}
\]

(A8)

we arrive at the Eqs. (38)–(40).

In the particular case of a perfectly reflecting plate in the retardation limit, it is convenient to replace the integration variable \(q\) in Eqs. (38)–(40) in favor of \(v = bc/u\), i.e., \(q = \sqrt{\nu^2 - 1} u/c\) [see Eq. (41)] and, hence,

\[
\int dq_{b} \ldots \rightarrow \int dv_{b} \ldots
\]

(A9)

For \(X \ll Z_s\), the exponential terms effectively limits the integrals in Eqs. (38)–(40) to the region where \(qX \ll 1\); hence, we can approximate \(J_v(qX)\) by \(J_v(0) \approx \delta_{v0}\), such that the nonzero scattering Green tensor components read

\[
G^{(1)}_{xx}(r_A, r_B, iu) = G^{(1)}_{yy}(r_A, r_B, iu)
\]

\[
= \frac{1}{8\pi Z_s} \left[ r_x - \left( 1 + \frac{c}{u Z_s} \right) + 2 \frac{c^2}{u^2 Z_s} \right] e^{i u Z_s/c}
\]

(A10)

\[
G^{(1)}_{zz}(r_A, r_B, iu) = -\frac{1}{2\pi Z_s} \left( \frac{c}{u Z_s} + \frac{c^2}{u^2 Z_s} \right) e^{-i u Z_s/c}
\]

(A11)

leading to Eqs. (47) and (48), recall Eq. (21).

In the nonretarded limit, it can be shown that the main contribution to the frequency integrals comes from the region where \(u/(cb) \ll 1\) (cf. [36]). In this region we have

\[
q = b \sqrt{1 - \frac{u^2}{b^2 c^2}} = b.
\]

(A12)

By changing the integration variable \(q\) according to

\[
\int dq_{b} \ldots \rightarrow \int db \ldots
\]

(A13)

and setting the lower limit of integration to zero, from Eqs. (38)–(40) we find, after some algebra, the nonzero elements of the scattering Green tensor to be approximately given by

\[
G^{(1)}_{xx}(r_A, r_B, iu) = \frac{2X^2 - Z_s^2 \frac{c^2}{2} r_p}{4\pi l_5^2 u^2}
\]

(A14)
with $l_s = \sqrt{X^2 + Z^2}$, leading to Eqs. (54) and (55).

For a magnetoelectric half space in the nonretarded limit, we apply a similar procedure as below Eq. (A11) and expand the reflection coefficients given by Eqs. (63) and (64) in terms of $u(bc)$,

$$r_s = \frac{\mu(iu) - 1}{\mu(iu) + 1} \frac{\mu(iu)[\epsilon(iu)\mu(iu) - 1]}{[\mu(iu) + 1]^2} \frac{u^2}{b^2 c^2}, \quad (A18)$$

$$r_p = \frac{\epsilon(iu) - 1}{\epsilon(iu) + 1} \frac{\epsilon(iu)[\epsilon(iu)\mu(iu) - 1]}{[\epsilon(iu) + 1]^2} \frac{u^2}{b^2 c^2}. \quad (A19)$$

Substituting (A18) and (A19) into Eqs. (38)–(40) and keeping only the leading-order terms of $u(bc)$, in the case of the purely electric half space we can ignore $r_s$ and the second term in the r.h.s. of Eq. (A19), so the relevant elements of the scattering Green tensor are approximately

$$G_{yy}^{(1)}(r_A, r_B, iu) = \frac{2X^2 - Z^2c^2 \epsilon(iu) - 1}{4\pi l_s^5 u^2 \epsilon(iu) + 1}, \quad (A20)$$

$$G_{yz}^{(1)}(r_A, r_B, iu) = -\frac{1}{4\pi l_s^5 u^2 \epsilon(iu) + 1}, \quad (A21)$$

$$G_{zx(zz)}^{(1)}(r_A, r_B, iu) = \frac{3XZc^2 \epsilon(iu) - 1}{4\pi l_s^5 u^2 \epsilon(iu) + 1}, \quad (A22)$$

$$G_{zz}^{(1)}(r_A, r_B, iu) = \frac{X^2 - 2Z^2c^2 \epsilon(iu) - 1}{4\pi l_s^5 u^2 \epsilon(iu) + 1}. \quad (A23)$$

For a purely magnetic half space, the first term on the r.h.s. of Eq. (A19) vanishes, so the leading order of $u(bc)$ is due to the second term as well as the first term on the r.h.s. of Eq. (A18); thus, the nonzero elements of the scattering Green tensor can be approximated by

$$G_{xx}^{(1)}(r_A, r_B, iu) = \frac{l_x - Z_s}{16\pi X^2} \mu(iu) - 1 \quad + \frac{Z_s l_s - Z^2_s}{16\pi X^2 l_s} [\mu(iu) - 1], \quad (A25)$$

$$G_{xx}^{(1)}(r_A, r_B, iu) = \frac{l_s}{16\pi l_s} \mu(iu) + 1, \quad (A26)$$

$$G_{zz}^{(1)}(r_A, r_B, iu) = \frac{1}{16\pi l_s} [\mu(iu) - 1]. \quad (A27)$$

### APPENDIX B

**Explicit Forms of $A_{ik}$ and $B_k$ in Eqs. (70) and (71)**

The integrals in Eqs. (70) and (71) can be performed to obtain the following explicit expressions:

$$A_{3+} = \frac{6\lambda}{(\lambda^2 + \zeta^2)^{5/2}}, \quad (B1)$$

$$A_{3-} = \frac{6(\lambda^3 - 4\lambda^2 \zeta^2)}{(\lambda^2 + \zeta^2)^{7/2}}, \quad (B2)$$

$$A_{3+} = \frac{3(4\lambda^2 - 3\lambda^2 \zeta^2)}{(\lambda^2 + \zeta^2)^{9/2}}, \quad (B3)$$

$$A_{3-} = \frac{30(4\lambda^5 - 41\lambda^3 \zeta^2 + 18\lambda \zeta^4)}{(\lambda^2 + \zeta^2)^{11/2}}, \quad (B4)$$

$$A_{3+} = \frac{3\lambda(2\lambda^2 - 3\zeta^2)}{(\lambda^2 + \zeta^2)^{11/2}}, \quad (B5)$$

$$B_3 = \frac{3\lambda(2\zeta^2 - 2\zeta^4)}{(\lambda^2 + \zeta^2)^{11/2}}, \quad (B6)$$

$$B_3 = \frac{15\lambda(8\lambda^4 - 40\lambda^2 \zeta^2 + 15\zeta^4)}{(\lambda^2 + \zeta^2)^{11/2}}. \quad (B7)$$

### ACKNOWLEDGMENTS

This work was supported by the Deutsche Forschungsgemeinschaft. H. S. would like to thank the Ministry of Science, Research, and Technology of Iran. H.T.D. is grateful to T. Kampf for a helpful hint on programming. He would also like to thank the Alexander
von Humboldt Stiftung and the National Program for Basic Research of Vietnam.

REFERENCES
34. W. C. Chew, Waves and Fields in Inhomogeneous Media (IEEE Press, N.Y., 1995), Secs. 2.1.3, 2.1.4, and 7.4.2.
35. M. Abramowitz and I. A. Stegun, Pocketbook of Mathematical Functions (Verlag Harri Deutsch, Frankfurt, 1984), Section 9.